(67) Evaluate $\lim_{x \to 0} \log_{\sin x} \sin 2x$.

Ans. → $\underset{x \to 0}{\text{Lt}} \log_{\sin x} \sin 2x$

Changing its base to e.

$$= \underset{x \to 0}{\text{Lt}} \log_e \sin 2x \times \log_{\sin x} e$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\log_e \sin 2x}{\log_e \sin x} \left[\frac{\infty}{\infty}\right]$$

Hence from L' Hospital's Rule, we have.

$$= \underset{x \to 0}{\text{Lt}} \frac{\frac{1}{\sin 2x} \times \cos 2x \times 2}{\frac{1}{\sin x} \times \cos x \times 1}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{2\cos 2x}{\sin 2x} \times \frac{\sin x}{\cos x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\cancel{2}\cos 2x \cdot \cancel{\sin x}}{\cancel{2}\cancel{\sin x} \cdot \cos x \cdot \cos x}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\cos 2x}{\cos x \cdot \cos x}$$

$$= \frac{\cos 0}{\cos 0 \times \cos 0} = \frac{1}{1 \times 1} = 1 \text{ Ans.}$$

(65) Evaluate $\underset{x\to 0}{Lt} \log_{\tan^2 x}(\tan^2 2x)$

Ans. → $\underset{x\to 0}{Lt} \log_{\tan^2 x}^{\tan^2 2x}$

Changing is base to e

$$= \underset{x\to 0}{Lt} \log_e^{\tan^2 2x} \times \log_{\tan^2 x}^{e}$$

$$= \underset{x\to 0}{Lt} \frac{\log_e^{\tan^2 2x}}{\log_e^{\tan^2 x}}$$

$$= \underset{x\to 0}{Lt} \frac{\cancel{2} \log_e^{\tan 2x}}{\cancel{2} \log_e^{\tan x}} \left[\frac{\infty}{\infty}\right]$$

Hence from L' Hospital's Rule; we have,

$$= \underset{x\to 0}{Lt} \frac{\frac{1}{\tan 2x} \times \sec^2 2x \times 2}{\frac{1}{\tan x} \times \sec^2 x \times 1}$$

$$= \underset{x\to 0}{Lt} \frac{\frac{\frac{2x\frac{1}{\cos^2 2x}\cos 2x}{2x}}{\frac{\sin 2x}{\cos 2x}}}{\frac{\frac{1}{\cos^2 x}\cos x}{\frac{\sin x}{\cos x}}} = \underset{x\to 0}{Lt} \frac{2\times\frac{1}{\cos 2x}\times\frac{1}{\sin 2x}}{\frac{1}{\cos x\cdot\sin x}}$$

$$= \underset{x\to 0}{Lt} \frac{2\sin x\cdot\cos x}{\sin 2x\cdot\cos 2x} = \frac{\cancel{\sin 2x}}{\cancel{\sin 2x}\times\cos 2x}$$

$$= \frac{1}{\cos 0} = \frac{1}{1} = 1 \text{ Ans.}$$

(67) Evaluate $\underset{x\to 0}{Lt} \frac{(1+x)^{\frac{1}{x}} - e}{x}$.

Ans. → $\underset{x\to 0}{Lt} \frac{(1+x)^{\frac{1}{x}} - e}{x}$

$$= \underset{x\to 0}{Lt} \frac{e^{\frac{1}{x}\log(1+x)} - e}{x}$$

$$= \underset{x\to 0}{Lt} \frac{e^{\left[\frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right]\right.} - e}{x}$$

$$= \underset{x\to 0}{Lt} \frac{e^{\frac{1}{\cancel{x}}\times\cancel{x}\left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right]} - e}{x}$$

$$= \underset{x\to 0}{Lt} \frac{e^{\left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right)} - e}{x}$$

$$= \underset{x \to 0}{Lt} \frac{e^{1} \times e^{-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots} - e}{x}$$

$$= \underset{x \to 0}{Lt} \frac{e \cdot e^{z} - e}{x} \qquad \left[\because z = -\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right]$$

$$= \underset{x \to 0}{Lt} \frac{e\left[\cancel{1} + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots - \cancel{1}\right]}{x}$$

Putting the value of z, we have;

$$= \underset{x \to 0}{Lt} \frac{e\left[\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right) + \frac{\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4}\right)^2}{2!} + \cdots\right]}{x}$$

$$= \underset{x \to 0}{Lt} \frac{e \times \cancel{x}\left[-\frac{1}{2} + \frac{x}{3} - \frac{x^2}{4} + \cdots\right] + \frac{x}{2!}\left(-\frac{1}{2} + \frac{x}{3} - \frac{x^2}{4}\right)^2 + \cdots}{\cancel{x}}$$

$$= -\frac{e}{2} \text{ Ans.}$$